Indexical Hybrid Tense Logic

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Abstract

In this paper we explore the logic of *now*, *yesterday*, *today* and *tomorrow* by combining the semantic approach to indexicality pioneered by Hans Kamp [9] and refined by David Kaplan [10] with hybrid tense logic. We first introduce a special *now* nominal (our $@_{now}$ corresponds to Kamp's original now operator N) and prove completeness results for both logical and contextual validity. We then add propositional constants to handle *yesterday*, *today* and *tomorrow*; our system correctly treats sentences like "Niels will die yesterday" as contextually unsatisfiable. Building on our completeness results for *now*, we prove completeness for the richer language, again for both logical and contextual validity.

Keywords: Hybrid logic, two-dimensional logic, nominals, indexicals, now

Human languages are rife with indexicals. Words like *now*, *here*, and *I* are context sensitive: when uttered at different times and places by different speakers they denote different times, places and people, and they do so in a constrained way. The indexical *now* picks out the time of utterance, *here* picks out the place, and *I* the speaker. These semantical constraints mean that they possess an interesting logic. The expression

I am here now,

for example, cannot be uttered falsely. We might not want to call it a logical validity, but it clearly is some kind of validity. We shall use the term *contextual validity* to distinguish such sentences from ordinary logical validities such as *Either it is raining or it is not.*

In this paper, which builds on Blackburn [2], we are going to examine the logic of four temporal indexicals, *now*, *yesterday*, *today* and *tomorrow* within the setting of hybrid tense logic. The distinguishing feature of hybrid tense logic is that it uses special propositional symbols called nominals to refer to times. A nominal i is true at a unique time; it names the time it is true at. We will treat *now* as a special nominal that is true at a contextually determined utterance time, and view *yesterday*, *today* and *tomorrow* as propositional constants true at unbroken stretches of time correctly aligned around this special point. The contextual semantics of these indexicals will be handled by the method pioneered by Hans Kamp in *Formal Properties of 'Now'* [9], and refined and extended by David Kaplan in *Demonstratives* [10]. Indeed, the present paper could be described as "Hybrid Logic meets Kamp-Kaplan semantics".

This meeting has two main advantages. The first is semantic. Hybridization enables us to perspicuously capture a number of important facts about indexicals and their interaction with the tenses. Consider, for example

Niels will die yesterday.

This is contextually incoherent. It cannot be truthfully uttered in any context because of the clash between the indexical *yesterday* (which places the dying in the past, namely sometime during yesterday) and the tensed verb *will die* which places the dying in the future. We shall represent this sentence as

$$F(\textit{yesterday} \wedge \textit{Niels-die})$$

and our semantics will guarantee that this formula cannot be satisfied in any context of utterance in any model, which is just as it should be.

The second advantage is logical. The literature on hybrid logic contains many general results on completeness and other topics; these results (which were proved for ordinary nominals) can be straightforwardly adapted to deal with the logic of *now*, and doing so sheds interesting light on results such as Kamp's eliminability result for his N operator. Moreover, once the logics of *now* have been captured, it is straightforward to build on them to capture the logics of *yesterday*, *today* and *tomorrow*. Indeed, by the end of the paper it should be clear that what we are presenting is not so much a particular logic of indexicality as a framework for modeling indexicality in a wide range of modal logics and languages.

1 Basic hybrid tense logic

The basic idea behind hybrid logic is to introduce a second sort of atomic symbol, nominals. Thus the point of departure for our work is a two-sorted language $\mathcal{L}$ which contains a countable set $\Phi = \{p, q, r, \ldots\}$ of propositional symbols and another (disjoint) countable set of nominals $\Omega = \{i, j, k, \ldots\}$. These are our atomic symbols. Our tense logic will be diamond-based: there is a diamond P for looking backwards in time, a diamond F for looking forwards, and for each nominal i an $@_i$-operator. Formulas of $\mathcal{L}$ are built as follows:

$$\phi ::= i \mid p \mid \perp \mid \neg\phi \mid \phi \wedge \psi \mid P\phi \mid F\phi \mid @_i\phi.$$

We define $H\phi$ to be $\neg P \neg\phi$ and $G\phi$ to be $\neg F \neg\phi$. A formula is said to be *pure* if its only atomic subformulas are nominals. Note that nominals can occur either as subscripts to @ ("in operator position") or as formulas in their own right ("in formula position").

The semantics for the basic hybrid tense logic is given by interpreting formulas of $\mathcal{L}$ in models based on a frame (T, R) together with a valuation V. Here T is a non-empty set and R is a binary relation. The elements of T are thought of as points of time and the relation R as the temporal flow. P searches backwards along this relation (into the past) whereas F searches forward (into the future). The valuation V distributes information over the frame, thus V takes atomic formulas to subsets of T and it satisfies the following two conditions:

(i) $V(p)$ is a subset of T, when $p \in \Phi$,

(ii) $V(i)$ is a singleton subset of T, when $i \in \Omega$.

We say that t is the denotation of i under V iff $t \in V(i)$.

Satisfiability in a model is defined in the usual way as a relation which obtains between a model $\mathfrak{M}$, a point t in the model, and a formula ϕ:

$$\begin{array}{ll}
\mathfrak{M}, t \models \bot & \text{never} \\
\mathfrak{M}, t \models a & \text{iff } a \text{ is atomic and } t \in V(a) \\
\mathfrak{M}, t \models \neg\phi & \text{iff } \mathfrak{M}, t \not\models \phi \\
\mathfrak{M}, t \models \phi \wedge \psi & \text{iff } \mathfrak{M}, t \models \phi \text{ and } \mathfrak{M}, t \models \psi \\
\mathfrak{M}, t \models P\phi & \text{iff for some } t',\ t'Rt \text{ and } \mathfrak{M}, t' \models \phi \\
\mathfrak{M}, t \models F\phi & \text{iff for some } t',\ tRt' \text{ and } \mathfrak{M}, t' \models \phi \\
\mathfrak{M}, t \models @_i\phi & \text{iff } \mathfrak{M}, t' \models \phi \text{ and } t' \in V(i).
\end{array}$$

A formula ϕ is *true in* $\mathfrak{M} = (T, R, V)$ when for all points $t \in T$ we have that $\mathfrak{M}, t \models \phi$. A formula is *logically valid* if it is true in all models.

The set of logically valid formulas (the basic hybrid tense logic) is called K_h^t. This logic can be proof theoretically characterized in a number of ways, for example via Hilbert systems [4], via natural deduction systems [6], or via tableau systems [3,5]. In this paper we shall use a tableau system; see the Appendix for details. For this system we have:

Theorem 1.1 (Basic Completeness) *Any set of formulas in $\mathcal{L}$ that is K_h^t-consistent is satisfiable in a model. Moreover, if Π is some set of pure $\mathcal{L}$-axioms, then any set of formulas which is $K_h^t + \Pi$-consistent is satisfiable in a model based on a frame satisfying the frame properties defined by Π.*

Note that this is not one, but many, completeness theorems. In our definition of the semantics, we imposed no restrictions on the relation R. However, given that we think of R as the flow of time, it would be natural to impose additional demands such as transitivity and irreflexivity. Theorem 1.1 tells us that if the required constraints can be expressed by a pure formula, then adding that formula as an axiom to K_h^t automatically yields completeness with respect to these properties. For example, $@_i\neg Fi$ and $FFi \to Fi$ express, respectively, the properties of irreflexivity and transitivity. Hence (as both formulas are pure) adding them as axioms to K_h^t yields a tableau system that is complete with respect to the class of frames in which R is a strict partial order.

2 Adding *now*

Our first step is to extend $\mathcal{L}$ with the atomic symbol *now* to obtain $\mathcal{L}(now)$. The new symbol is in essence a nominal, but it is a very special one with a very special (indexical) meaning. So with the introduction of *now* we have three sorts of atomic formulas: the propositional symbols Φ, the ordinary nominals Ω, and the indexical *now*. The formulas of $\mathcal{L}(now)$ are built as follows:

$$\phi ::= now \mid i \mid p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid P\phi \mid F\phi \mid @_{now}\phi \mid @_i\phi.$$

Models for $\mathcal{L}(now)$ will be contextualized versions of the ordinary models presented in the previous section. A contextual model $\mathfrak{M}$ is a 5-tuple

$$\mathfrak{M} = (T, R, V, C, \eta)$$

where (T, R) is an ordinary frame, V is a valuation function (to be defined below), C is a non-empty set of contexts, and η is a mapping from contexts to points in T. The function η is crucial: it specifies, for any context $c \in C$, what the time (or temporal location) of any utterance in that context is. That is, it tells you, for any context, what your "now" moment is. Thus η is exactly what Kaplan [10] calls the *character* of "now". As before, the valuation V interprets atomic formulas, but now it does so relative to contexts. Therefore a contextual valuation V takes a pair (c, a) consisting of a context c and an atom a and assigns a subset of T to the pair subject to the following restrictions:

(i) $V(c, p)$ is a subset of T, when $p \in \Phi$,

(ii) $V(c, i)$ is a singleton subset of T, when $i \in \Omega$,

(iii) $V(c, now) = \{\eta(c)\}$.

Note the restriction hard-wired into the semantics of *now*. This special nominal denotes a singleton, but not just any singleton: in any context c it denotes the utterance time $\eta(c)$ in that context. Satisfiability in contextual models is construed as a relation obtaining between four elements: a model $\mathfrak{M} = (T, R, V, C, \eta)$, a context c, a point t, and a formula ϕ:

$\mathfrak{M}, c, t \models \bot$	never
$\mathfrak{M}, c, t \models a$	iff a is atomic and $t \in V(c, a)$
$\mathfrak{M}, c, t \models \neg\phi$	iff $\mathfrak{M}, c, t \not\models \phi$
$\mathfrak{M}, c, t \models \phi \wedge \psi$	iff $\mathfrak{M}, c, t \models \phi$ and $\mathfrak{M}, c, t \models \psi$
$\mathfrak{M}, c, t \models P\phi$	iff for some t', $t'Rt$ and $\mathfrak{M}, c, t' \models \phi$
$\mathfrak{M}, c, t \models F\phi$	iff for some t', tRt' and $\mathfrak{M}, c, t' \models \phi$
$\mathfrak{M}, c, t \models @_a\phi$	iff $\mathfrak{M}, c, t' \models \phi$ and $t' \in V(c, a)$, where a is *now* or $a \in \Omega$.

Let's think carefully about what the last clause means when working with the $@_{now}$-operator. When we evaluate $@_{now}\phi$ at (c, t) (that is, when we ask whether $\mathfrak{M}, c, t \models @_{now}\phi$ holds) we jump to the point $\eta(c) \in T$ associated with c and ask whether ϕ is satisfied there relative to context c (recall the intuition that $\eta(c)$ is the temporal location of any utterance in c). That is:

$$\mathfrak{M}, c, t \models @_{now}\phi \quad \text{iff} \quad \mathfrak{M}, c, \eta(c) \models \phi.$$

What about validity? Well, clearly we can generalize our previous definition. We say that ϕ is *true in* $\mathfrak{M}$ if for all pairs c, t from $\mathfrak{M}$ we have $\mathfrak{M}, c, t \models \phi$. And then we say that ϕ is *logically valid* if ϕ is true in all models. This is essentially the regular notion of validity familiar from modal logic.

But the whole point of working with contextualized models is that they support a second notion of validity, *contextual validity.* We are interested in characterising not only the set of logically valid formulas, but also *the set of formulas that are always true whenever they are uttered.* To put this more precisely: we say, with respect to $\mathfrak{M}$, that ϕ *is satisfied in the context* c precisely when ϕ in $\mathfrak{M}$ is satisfied in c at the utterance time of c, that is, whenever

$$\mathfrak{M}, c, \eta(c) \models \phi.$$

We say that ϕ is *contextually true in* $\mathfrak{M}$ when ϕ is satisfied in every context c in $\mathfrak{M}$. And ϕ is *contextually valid* if it is contextually true in every model. [1]

Let us consider some examples of contextual validities. Clearly ordinary tautologies from propositional logic are logically valid. And trivially, any formula which is logically valid is also contextually valid. On the other hand, *now* is not logically valid, but it is contextually valid: given any $\mathfrak{M}$, and any c, $\eta(c)$ is the denotation of *now* under V in c, That is, for any context c, $\eta(c) \in V(c, now)$ and so we always have that:

$$\mathfrak{M}, c, \eta(c) \models now$$

In Kaplan's words: *now* "cannot be uttered falsely" (Kaplan, 1978, p. 402).

Now for two simple but important model-theoretic properties of contextual semantics with respect to $\mathcal{L}$ (recall that $\mathcal{L}$ is $\mathcal{L}(now)$ without *now*):

Lemma 2.1 (Playing with Contexts) *All $\mathcal{L}$-formulas are semantically insensitive to contexts:*

(i) *Any ordinary model can be extended to a contextual model which, with respect to $\mathcal{L}$-formulas, is semantically equivalent to the original model.*

(ii) *Moreover, $\mathcal{L}$-formulas are insensitive to how contexts are mapped to moments of time.*

Proof. Part (i). Let $\mathfrak{M}$ be an ordinary model (T, R, V) and let $\mathfrak{M}'$ be $\mathfrak{M}$ contextualised by $C = \{c\}$ and η defined by $\eta(c) = t_1$, where t_1 is the denotation of i_1 (the first nominal in some enumeration of the nominals) under V. Moreover, let V' be defined by $V'(c, a) = V(a)$ for any atomic a. With $\mathfrak{M}' = (T, R, V', C, \eta)$ we have for any formula ϕ of $\mathcal{L}$ that:

$$\mathfrak{M}, t \models \phi \quad \text{iff} \quad \mathfrak{M}', c, t \models \phi.$$

[1] Contextually validity is what Kaplan calls *validity* [10, p. 547]. This strikes us as confusing. We prefer to use the more explicit *contextual validity* to clearly signal when we are talking about the Kamp-Kaplan notion of validity, and *logical validity* for regular modal validity.

The proof is a trivial induction on the complexity of ϕ.

Part (ii). Given any model $\mathfrak{M} = (T, R, V, C, \eta)$ we can replace η with an arbitrary η' having C as domain, resulting in a model $\mathfrak{M}' = (T, R, V', C, \eta')$, where V' is defined by the following:

$$V'(c, a) = \begin{cases} \{\eta'(c)\}, & \text{if } a \text{ is } now, \\ V(c, a), & \text{otherwise.} \end{cases}$$

Then $\mathfrak{M}$ and $\mathfrak{M}'$ have the same semantical behaviour with respect to $\mathcal{L}$-formulas:

$$\mathfrak{M}, c, t \models \phi \quad \text{iff} \quad \mathfrak{M}', c, t \models \phi.$$

Once again, the proof is a trivial induction on the structure of $\mathcal{L}$-formulas. $\Box$

To conclude this section we briefly discuss Hans Kamp's [9] eliminability result for his now operator N (for a succinct overview, see Burgess [7]). Expressed in our notation, Kamp's result is for the language of tense logic together with $@_{now}$ (that is, his language contains no ordinary nominals, no ordinary @-operators, and no *now* nominal). He proved that if a formula containing occurrences of $@_{now}$ is satisfiable at the designated now point (that is, $\eta(c)$), then there is an equivalent $@_{now}$ free formula satisfiable at that point.

Kamp's result can be viewed as a special case of a more general observation, due to Balder ten Cate [8], concerning arbitrary @-operators. Let's first consider the special case of Kamp's language. Suppose that ϕ is a formula in the restricted language just described, and suppose that $@_{now}\psi$ is some subformula occurrence in ϕ. Then (following ten Cate) we observe that:

$$\big(@_{now}\psi \wedge \phi[@_{now}\psi \leftarrow \top]\big) \vee \big(\neg @_{now}\psi \wedge \phi[@_{now}\psi \leftarrow \bot]\big)$$

is equivalent to ϕ. By iteratively continuing this process we eventually arrive at a formula where no occurrence of $@_{now}$ lies in the scope of any other $@_{now}$ or any tense operator. This longer formula (call it ϕ^+) is equivalent to ϕ, and hence satisfiable at the utterance time iff ϕ is. And now a simple observation yields Kamp's result: at the utterance time, for any formula θ we have $@_{now}\theta \leftrightarrow \theta$. So replacing each subformula of the form $@_{now}\theta$ in ϕ^+ by θ, yields an equivalent $@_{now}$-free formula ϕ^k which is satisfiable at the utterance time iff ϕ^+ is.

That gives us the classic Kamp result, but ten Cate's simple argument works for *arbitrary* $@_i$-operators, not just for the special case of $@_{now}$. So it is easy to generalize Kamp's result to the full language of this paper. Suppose we have a formula ϕ of this language. Following ten Cate, we expand our original ϕ to an equivalent ϕ^+ with the property that no $@_i$-operator or $@_{now}$ lies under the scope of any other @-operator, or $@_{now}$, or any tense operator. And then we argue as above: any $@_i$-operator can be eliminated at the denotation of i using the equivalence $@_i\phi \leftrightarrow \phi$. Note that this argument does *not* eliminate nominals (including *now*) that occur in formula position; it simply shows that we can select any operator $@_i$ and eliminate it at the denotation of i.

3 Completeness for *now*

In this section we are going to look at the two languages $\mathcal{L}$ and $\mathcal{L}(now)$ with respect to the two different notions of validity given by contextual semantics. That is, we are going to examine four logics and establish the results given in the following matrix:

	$\mathcal{L}$	$\mathcal{L}(now)$
Logical validity	**Logic 1:** K_h^t	**Logic 3:** $K_h^t(now)$
Contextual validity	**Logic 2:** K_h^t	**Logic 4:** $K_h^t(now) + now$

Recall that K_h^t is the basic hybrid tense logic in language $\mathcal{L}$. The logic $K_h^t(now)$ is obtained by applying the tableau rules of K_h^t to the formulas of $\mathcal{L}(now)$ treating *now* as if it were an ordinary nominal. We'll explain what $K_h^t(now) + now$ is later in this section.

Our arguments will be semantic. In essence, in the work that follows we are trying to pin down when *now* behaves like a regular nominal, and when it does not (that is, when its special contextual properties kick in). This information will be valuable when we later add *yesterday*, *today* and *tomorrow*. As a preliminary step, we state the soundness of our tableau system.

A formula is called a *satisfaction statement* if it is either of the form $@_a\phi$ or $\neg@_a\phi$, where a is *now* or an ordinary nominal. If Σ is a set of satisfaction statements and $\mathfrak{M}$ is a contextual model, then we say that Σ is *satisfied by label* in $\mathfrak{M}$ if there is a context c such that for every formula in Σ:

(i) If $@_a\phi \in \Sigma$ then $\mathfrak{M}, c, \overline{a} \models \phi$,

(ii) If $\neg@_a\phi \in \Sigma$ then $\mathfrak{M}, c, \overline{a} \not\models \phi$,

where $\overline{a}$ is the denotation of a under V in c.

Lemma 3.1 (Soundness) *For any set of satisfaction statements Σ in $\mathcal{L}(now)$, if Σ is satisfiable by label, then so is at least one of the sets obtained by applying any rule of $K_h^t(now)$ to Σ. Hence both $K_h^t(now)$ and its subsystem K_h^t are sound.*

Proof. Proving this requires essentially nothing beyond the soundness proof for ordinary semantics given in Blackburn [3]. The basic point is that as far as tableaux rules are concerned, *now* behaves much like an ordinary nominal. □

Logically and contextually valid formulas in $\mathcal{L}$

Let's turn to completeness. Our first result should not be a surprise. It says that K_h^t captures logical validity in $\mathcal{L}$ with respect to contextual semantics.

Theorem 3.2 *K_h^t is complete with respect to logically valid $\mathcal{L}$-formulas.*

Proof. Let ϕ be some K_h^t-consistent $\mathcal{L}$-formula. By Theorem 1.1, ϕ is satisfiable in some ordinary model. But then, by Lemma 2.1, ϕ is satisfiable in a contextual model too. □

But we can also show that, for $\mathcal{L}$, K_h^t is still the complete tableau system when we turn from logical validity to contextual validity (recall that a formula ϕ is contextually valid iff for all $\mathfrak{M}$ and all $c, \mathfrak{M}, c, \eta(c) \models \phi$). And this should not be surprising either—after all, without *now* in the language, we have nothing that gets to grips with contexts. And that's indeed how things turn out:

Theorem 3.3 K_h^t *is complete with respect to contextually valid* $\mathcal{L}$*-formulas.*

Proof. We show that the set of contextually valid $\mathcal{L}$-formulas equals the set of logically valid $\mathcal{L}$-formulas. Given that, the result follows from Theorem 3.2.

Suppose for the sake of contradiction that ϕ in $\mathcal{L}$ is contextually valid but not logically valid. Then there is a model $\mathfrak{M} = (T, R, V, C, \eta)$, a context c, and a point t such that $\mathfrak{M}, c, t \not\models \phi$. Define $\eta' : C \to T$ to be constantly t and replace η in the definition of V by η' to obtain $\mathfrak{M}' = (T, R, V', C, \eta')$. As formulas in $\mathcal{L}$ are insensitive to how contexts are mapped to moments in time (Lemma 2.1.ii) we have $\mathfrak{M}', c, \eta'(c) \not\models \phi$ which contradicts the contextual validity of ϕ. □

Summing up, we have established the results in the left-hand column of our matrix: K_h^t is both **Logic 1** and **Logic 2**. Time to turn to $\mathcal{L}(now)$.

Logically and contextually valid formulas in $\mathcal{L}(now)$

The tableau system $K_h^t(now)$ in $\mathcal{L}(now)$—where *now* is treated as an ordinary nominal—will characterise the logically valid $\mathcal{L}(now)$-formulas. The contextually valid formulas, however, will be characterised by $K_h^t(now)$ together with the formula *now* regarded as a single axiom; we call the tableau system embodying this idea $K_h^t(now) + now$. More precisely, an $\mathcal{L}(now)$-formula ϕ is provable in $K_h^t(now) + now$ if and only if there is a finite closed tableau whose root node is $@_i(now \wedge \neg\phi)$ where i does not occur in ϕ. The idea is to enforce i to denote both the utterance time and the point where ϕ must be falsified; if we can't falsify ϕ at the utterance time then it must be contextually valid.[2] In order to prove that these two systems characterise the two different notions of validities we need some technicalities regarding substitutions. Moreover, until now, we have worked with fixed sets of propositional symbols Φ and nominals Ω, but in the proofs that follow we shall need a nominal not in Ω. Thus, for the rest of this section, let j be a fresh nominal not in this set, and let our tableau system in $\mathcal{L}$ extended with nominal j be called $K_h^t(j)$. Finally, we'll use the following notation: if in ϕ we uniformly substitute ρ for ψ we obtain $\phi[\psi \leftarrow \rho]$.

Lemma 3.4 (Playing with Substitutions) *With regard to satisfiability and consistency, now is just a nominal:*

(i) *If ϕ in $\mathcal{L}$ is satisfiable at a pair (c, t) from a model $\mathfrak{M}$, then now can be uniformly substituted for any nominal i in ϕ and the resulting formula $\phi[i \leftarrow now]$ is also satisfiable at (c, t) at a model $\mathfrak{M}'$ which differs from $\mathfrak{M}$ only in the value it assigns to now.*

(ii) *Given any formula ϕ of $\mathcal{L}(now)$, ϕ is $K_h^t(now) + now$-consistent if and only if $j \wedge \phi[now \leftarrow j]$ is $K_h^t(j)$-consistent.*

[2] In practice, when proving formulas in $K_h^t(now) + now$ we build a tableau whose root node is $@_i now$ and whose second node is $\neg @_i \phi$, thereby saving two steps.

(iii) *Given any formula* ϕ *of* $\mathcal{L}(now)$*, if* $\mathfrak{M}, c, \eta(c) \models j \wedge \phi[now \leftarrow j]$ *then* $\mathfrak{M}, c, \eta(c) \models now \wedge \phi$.

Proof. Part (i). Suppose ϕ in $\mathcal{L}$ is satisfiable in $\mathfrak{M} = (T, R, V, C, \eta)$ and suppose i is a nominal in ϕ. As ϕ is insensitive to contexts we simply take some context c and ask what the denotation of i under V in context c is. Suppose it is t'. We let η' be constantly t' and let V' just be V with η replaced by η'. For $\mathfrak{M}' = (T, R, V', C, \eta')$ it can be proved by an easy induction that for any $t \in T$ and any subformula ψ of ϕ:

$$\mathfrak{M}, c, t \models \psi \quad \text{iff} \quad \mathfrak{M}', c, t \models \psi[i \leftarrow now].$$

Part (ii). We first note two simple but useful facts:

- For any ψ in $\mathcal{L}(now)$ the formula $\psi[now \leftarrow j]$ is in $\mathcal{L}(j)$ and j only occurs where it has replaced *now*.
- If ψ is a formula in $\mathcal{L}(j)$ then $\psi[j \leftarrow now]$ is in $\mathcal{L}(now)$, and *now* occurs only in $\psi[j \leftarrow now]$ where it has replaced j.

Hence part (ii) follows easily. For suppose $j \wedge \phi[now \leftarrow j]$ is $K_h^t(j)$-consistent. Then any tableau having $@_i(j \wedge \phi[now \leftarrow j])$ as root node will contain an open branch. Therefore there cannot be a closed tableau in $K_h^t(now) + now$ with $@_i(now \wedge \phi)$ at the root, as this could be turned into a closed tableau for $@_i(j \wedge \phi[now \leftarrow j])$ by uniformly substituting j for *now*. And conversely, if ϕ in $\mathcal{L}(now)$ is $K_h^t(now) + now$-consistent, then any tableau for $@_i(now \wedge \phi)$ will contain an open branch. So there cannot be closed tableau in $K_h^t(j)$ with $@_i(j \wedge \phi[now \leftarrow j])$ at the root either.

Part (iii). Given the assumption that $\mathfrak{M}, c, \eta(c) \models j \wedge \phi[now \leftarrow j]$ it is easy to prove, that for any subformula ψ of ϕ

$$\mathfrak{M}, c, \eta(c) \models \psi[now \leftarrow j] \quad \text{iff} \quad \mathfrak{M}, c, \eta(c) \models \psi.$$

For the atomic case the claim is completely trivial for all atomic formulas except when ψ is *now*. But then $\mathfrak{M}, c, \eta(c) \models j$ iff $\mathfrak{M}, c, \eta(c) \models now$ as we are working under the assumption that j denotes $\eta(c)$ under V in context c. The induction step then follows trivially except for when ψ is $@_{now}\theta$. But again, this follows by assumption as j and *now* both denote $\eta(c)$ under V in context c. □

Theorem 3.5 $K_h^t(now)$ *is complete with respect to the logically valid formulas in contextual semantics.*

Proof. Let ϕ be a formula of $\mathcal{L}(now)$ and suppose ϕ is $K_h^t(now)$-consistent. Let i be the first nominal not occurring in ϕ and substitute i for *now* in ϕ. Clearly $\phi[now \leftarrow i]$ is $K_h^t(now)$-consistent, thus by Theorem 3.2 $\phi[now \leftarrow i]$ has a model. We then apply Lemma 3.4.i to get the required model for ϕ. □

Theorem 3.6 $K_h^t(now)$+*now is complete with respect to the contextually valid formulas in contextual semantics.*

Proof. Suppose ϕ in $\mathcal{L}(now)$ is $K_h^t(now) + now$-consistent. By Lemma 3.4.ii, $j \wedge \phi[now \leftarrow j]$ is $K_h^t(j)$-consistent, hence by Theorem 3.2 there is a model $\mathfrak{M} = (T, R, V, C, \eta)$ such that:

$$\mathfrak{M}, c, t \models j \wedge \phi[now \leftarrow j].$$

As $\phi[now \leftarrow j]$ does not contain *now*, it is insensitive to the way C is mapped on T (Lemma 2.1.ii). We then do the usual trick of defining η' to be constantly t, and this induces the usual V' obtained from V by replacing η by η'. We let $\mathfrak{M}' = (T, R, V', C, \eta')$ and obtain:

$$\mathfrak{M}', c, \eta'(c) \models j \wedge \phi[now \leftarrow j].$$

By Lemma 3.4.iii, $now \wedge \phi$, and thus ϕ, is satisfiable at $c, \eta'(c)$ in $\mathfrak{M}'$ too. □

We end our discussion of *now* with a general theorem.

Theorem 3.7 *Let Λ be K_h^t extended with pure axioms, and $\Lambda + now$ be its contextualized counterpart. Then:*

(i) *$\Lambda + now$ is contextually complete with respect to the same classes of models as that Λ is logically complete for.*

(ii) *Λ-satisfiability has the same complexity as $\Lambda + now$ satisfiability.*

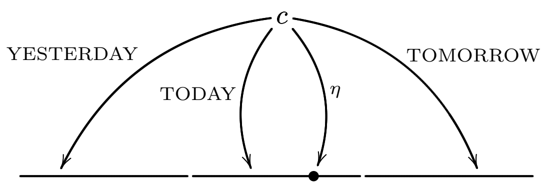
(iii) *There is a terminating tableau system for $\Lambda + now$ iff there is a terminating tableau system for Λ.*

Proof. Part (i) is essentially the standard result, Theorem 1.1, from hybrid logic: pure axioms restrict us to the appropriate model classes, and adding *now* does not effect this, as satisfiability in a contextual model for any ϕ in $\mathcal{L}(now)$ can be reduced to satisfiability of $\phi[now \leftarrow j]$ in an *ordinary* model (where j is fresh). As for Parts (ii) and (iii), it follows by our results above that finding a satisfying model (or a terminating tableau) for $@_i(now \wedge \phi)$ amounts to finding a satisfying model (or a terminating tableau) for $@_j(j \wedge \phi[now \leftarrow j])$, where j is a fresh nominal. □

4 Adding *yesterday*, *today* and *tomorrow*

Let's extend $\mathcal{L}(now)$ with three symbols: *yesterday*, *today* and *tomorrow*. Like *now* these are indexicals, but they are not nominals. Rather, they are special propositional symbols: each denotes a "daylike" set of points correctly positioned in the model with respect to the utterance time. So the formulas of $\mathcal{L}(now, yesterday, today, tomorrow)$ are formed like those of $\mathcal{L}(now)$, except that we can use *yesterday*, *today* and *tomorrow* as atomic symbols. It's perhaps worth emphasizing that these three new symbols only occur in formula position, never in operator position; they are not nominals, and @ requires nominals as subscripts.

What about models? The key point is to add further structure to contexts. Until now, the only structure on contexts has been the function η which returns the utterance time. Here we will add three further functions: YESTERDAY, TODAY, and TOMORROW. These map contexts to sets of times, and we impose a number of constraints. These constraints ensure that the three sets are correctly "contextually placed", with respect to the utterance time $\eta(c)$ and each other, and that they are sufficiently "daylike". In essence we are specifying what Kaplan calls the character of *yesterday*, *today* and *tomorrow*, and the following diagram shows what we require:



An important remark: we are *not* going to impose any global requirements on R. It would be easy to insist that R be irreflexive, transitive, or linear, but we won't do this. Instead, we impose structure *locally*, that is, on only these three sets of times. Why work this way? For a number of reasons. For a start, working locally means that our approach can be used with very weak tense logics. Moreover, there is no one class of temporal models suitable for every application: philosophers may be torn between Ockhamist and Peircean branching time, semanticists may demand linear time, while computer scientists may want discrete time for some applications and dense or even continuous time for others. We want our analysis to adapt easily to all such demands.

Let's make the pictorial constraints on character explicit. For all $c \in C$, YESTERDAY(c), TODAY(c) and TOMORROW(c) are subsets of T such that:

(i) $\eta(c) \in$ TODAY(c).

(ii) $\eta(c)$ is an R-successor of every point in YESTERDAY(c).

(iii) $\eta(c)$ is an R-predecessor of every point in TOMORROW(c).

(iv) YESTERDAY(c), TODAY(c), and TOMORROW(c) are pairwise disjoint.

(v) Every point in YESTERDAY(c) R-precedes every point in TODAY(c).

(vi) Every point in TODAY(c) R-precedes every point in TOMORROW(c).

(vii) Every point in YESTERDAY(c) R-precedes every point in TOMORROW(c).

(viii) YESTERDAY(c), TODAY(c), and TOMORROW(c) are all R-convex.

(ix) If $t \in$ YESTERDAY(c) and $t' \in$ TODAY(c) and tRs and sRt', then either $s \in$ YESTERDAY(c) or $s \in$ TODAY(c).

(x) If $t \in$ TODAY(c) and $t' \in$ TOMORROW(c) and tRs and sRt', then either $s \in$ TODAY(c) or $s \in$ TOMORROW(c).

In the presence of global assumptions about R (such as irreflexivity and transitivity) this list contains redundancies; for example, we don't need item vii if R is transitive. Its virtue (as we shall see below) is that even in the absence of such assumptions it imposes enough constraints to support interesting local inferences about contextuality.

Models for our expanded language simply build in this extra structure. That is, a contextual model is now an 8-tuple

$$\mathfrak{M} = (T, R, V, C, \eta, \textsc{yesterday}, \textsc{today}, \textsc{tomorrow}),$$

and it only remains to specify the valuation for our three new atomic symbols:

(i) $V(c, \mathit{yesterday}) = \textsc{yesterday}(c)$,

(ii) $V(c, \mathit{today}) = \textsc{today}(c)$,

(iii) $V(c, \mathit{tomorrow}) = \textsc{tomorrow}(c)$.

So what about completeness? Let us us first deal with logical validity. We define $K_h^t(\mathit{now}, \mathit{yesterday}, \mathit{today}, \mathit{tomorrow})$ to be $K_h^t(\mathit{now})$ augmented with all instances of the following axioms. That is, we extend our tableau system by allowing any of the following formulas to be freely introduced in the course of tableau construction:

Now Placement	**Disjointness**
$\mathit{now} \to \mathit{today}$	$\mathit{today} \to \neg \mathit{tomorrow}$
$\mathit{yesterday} \to F\mathit{now}$	$\mathit{today} \to \neg \mathit{yesterday}$
$\mathit{tomorrow} \to P\mathit{now}$	$\mathit{yesterday} \to \neg \mathit{tomorrow}$
One Step Alignment	**Two Step Alignment**
$\mathit{today} \to G\neg \mathit{yesterday}$	
$\mathit{tomorrow} \to G\neg \mathit{today}$	$\mathit{tomorrow} \to G\neg \mathit{yesterday}$
Convexity	**No Gaps**
$P\mathit{yesterday} \wedge F\mathit{yesterday} \to \mathit{yesterday}$	$P\mathit{yesterday} \wedge F\mathit{today} \to \mathit{yesterday} \vee \mathit{today}$
$P\mathit{today} \wedge F\mathit{today} \to \mathit{today}$	$P\mathit{today} \wedge F\mathit{tomorrow} \to \mathit{today} \vee \mathit{tomorrow}$
$P\mathit{tomorrow} \wedge F\mathit{tomorrow} \to \mathit{tomorrow}$	

These axioms correspond in an obvious way to the ten requirements we demand of our character functions. Indeed—because of the clarity of the correspondences involved—it would be straightforward to impose even *less* structure simply by dropping suitable axioms. For example, for some applications we might want to think of all three days as points, in which case we would simply drop the Convexity and No Gaps axioms. But in any case, adding all the above axioms results in *logical* completeness with respect to context structures as defined by the ten constraints on character.

But now for the key issue: how do we get *contextual* completeness? Exactly as we did before. Let

$$K_h^t(\mathit{now}, \mathit{yesterday}, \mathit{today}, \mathit{tomorrow}) + \mathit{now}$$

be $K_h^t(\mathit{now}, \mathit{yesterday}, \mathit{today}, \mathit{tomorrow})$ extended by insisting that the tableau construction for a formula ϕ starts with $@_k(\mathit{now} \wedge \phi)$ at the root node, or equivalently, with $@_k \mathit{now}$ at the root node and $@_k\phi$ immediately afterwards (as before, k is a nominal not occuring in ϕ). Informally, just as with $K_h^t(now)+now$, contextual validity is captured by asserting *Now!* at the start of tableaux construction. And a little reflection shows why this must be so. If we utter *Now!* in a context, then modus ponens fires the Now Placement axioms and we immediately infer that *It's Today!*, that *It's not Yesterday!* and that *It's not Tomorrow!* And (just glance through the axiom list) modus ponens and modus tollens keep firing until we have full information about the relative location and structure of the three days. To put it another way: *now* is the only bridge we require between realms of logical and contextual validity. Uttering *now* nails $\eta(c)$ firmly into the temporal flow, and *yesterday*, *today*, and *tomorrow* line up obediently around it.

Here is an example of the system in action. We shall show that in any context of utterance, it is impossible to satisfy

$$F(yesterday \wedge Niels\text{-}die).$$

Here is the required tableau:

1 $@_k now$	
2 $@_k F(yesterday \wedge Niels\text{-}die)$	
3 $@_k Fi$	$2, F$ *Elimination*
4 $@_i (yesterday \wedge Niels\text{-}die)$	$2, F$ *Elimination*
5 $@_i yesterday$	$4, \wedge$ *Elimination*
6 $@_i Niels\text{-}die$	$4, \wedge$ *Elimination*
7 $@_k (now \rightarrow today)$	*Now Placement axiom*
8 $@_k today$	$1, 7,$ *Modus Ponens*
9 $@_k (today \rightarrow G\neg yesterday)$	*One Step Alignment axiom*
10 $@_k G\neg yesterday$	$8, 9,$ *Modus Ponens*
11 $@_i \neg yesterday$	$3, 10,$ *G Propogation*
12 $\perp$	$5, 11$

Theorem 4.1 *Let Λ be $K_h^t(now, yesterday, today, tomorrow)$ extended with pure axioms, and let $\Lambda+now$ be its contextualized counterpart. Then $\Lambda+now$ is contextually complete with respect to the same class of models that Λ is logically complete for.*

Proof. As we have already informally discussed, beginning the tableau construction with $@_k now$ and $\neg @_k \phi$ is all that needs to be done to test ϕ for contextual rather than logical validity. The claim about completeness for pure logics Λ is just the familiar point that properly designed hybrid proof systems are complete for any pure extension of the minimal logic. □

But the previous theorem is only a partial analog of Theorem 3.7. In our earlier work on $K_h^t(now) + now$ we obtained not merely general completeness results but also general results concerning complexity and tableau termination. With the richer language, however, matters are more complicated: the computational consequences of our use of axioms for character is unclear. But for some classes of commonly used models (such as those based on linear frames) more can be said. Here's an example.

Nowadays modal logicians often add A, the universal modality, as a new primitive, to logics they find interesting: $A\phi$ asserts that ϕ holds at *all* points in a model. But on some classes of model A may be definable using the tense operators. For example, when working with transitive linear models, $A\phi$ is definable as $H\phi \vee \phi \vee G\phi$. This leads to the following:

Theorem 4.2 *Let* M *be a class of models on which the universal modality is definable in terms of the tense operators. Then the complexity of the satisfiability problem for* $K_h^t(now, yesterday, today, tomorrow)$ *is the same as the complexity satisfiability problem for* K_h^t *over* M*, and so is the satisfiability problem for* $K_h^t(now, yesterday, today, tomorrow) + now$*.*

Proof. Let's first treat $K_h^t(now, yesterday, today, tomorrow)$ satisfiability. Let **Character** be the conjunction of the character axioms listed above. Then observe that $K_h^t(now, yesterday, today, tomorrow)$ satisfying a formula ϕ is the same as satisfying the conjunction

$$\phi \wedge A(\mathbf{Character}).$$

That is, we are relying on the universal modality to appropriately constrain the denotations of the day-indexicals, and clearly it is strong enough do this.

Now, by assumption, A is definable using the tense operators, but we said nothing about how complex this definition was (some definitions might lead to blowups in formula size). But this is irrelevant: **Character** is a fixed formula, thus $A(\mathbf{Character})$ has constant size, and so we have that the length of $\phi \wedge A(\mathbf{Character})$ depends only on the length of ϕ. So we have polynomial time reduced satisfiability for $K_h^t(now, yesterday, today, tomorrow)$ to satisfiability for $K_h^t(now)$, and (by replacing all occurrences of now and $@_{now}$ using some fresh nominal k, as in the proof of Theorem 3.7) this in turn reduces essentially to K_h^t satisfiability over M.

As for $K_h^t(now, yesterday, today, tomorrow) + now$ satisfiability, here the task is to satisfy ϕ in some model at the utterance time. But this simply means: find a model for $now \wedge \phi$. But this is a $K_h^t(now, yesterday, today, tomorrow)$ satisfiability task. □

5 Concluding Remarks

In this paper we have shown how to incorporate the temporal indexicals *now*, *yesterday*, *today* and *tomorrow* into hybrid tense logic and have provided completeness theorems for both ordinary logical validity and the Kamp-Kaplan notion of contextual validity. Our analyses have been modular. Perhaps most importantly, the bridge between logical and contextual validity has been provided in a uniform way for both the *now* language, and the stronger language containing *yesterday*, *today* and *tomorrow*; in both cases, simply adding the utterance *now* as an additional contextual axiom is enough to lift us from logical to contextual validities. Furthermore, the local approach we adopted when introducing the character constraints on *yesterday*, *today* and *tomorrow* means that our results are compatible with a wide range of global assumptions about the structure of time. Finally, the fact we are working in hybrid logic means that our tableau systems can easily be strengthened (with the help of pure axioms) from the minimal tense logic (no conditions on R) to more temporally interesting classes of models, such as strict partial orders or strict total orders—or, indeed, even to the richer setting of interval-based logic.

But in our view hybrid logic has helped in a less obvious but no less important way. We believe that some work in the Kamp-Kaplan tradition has gone astray by using contextual semantics to simulate temporal reference in a rather artificial way; see van Benthem [1] for a detailed critique of this tendency. Using hybrid logic avoids this pitfall: it provides the tools required to tackle temporal reference head on. But this means that we can use Kamp-Kaplan semantics for the purpose for which it was originally intended, and for which it is best suited: *handling indexicality.* In our view, hybrid logic is capable of bringing clarity and simplicity to this interesting but complex area, and we hope to substantiate this claim in future work.

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References

[1] Benthem, J. V., *Tense logic and standard logic*, Logique et Analyse **20** (1977), pp. 395–437.
[2] Blackburn, P., *Tense, Temporal Reference and Tense Logic*, Journal of Semantics **11** (1994), pp. 83–101.
[3] Blackburn, P., *Internalizing labelled deduction*, Journal of Logic and Computation **10** (2000), pp. 137–168.
[4] Blackburn, P. and B. T. Cate, *Pure Extensions, Proof Rules, and Hybrid Axiomatics*, Studia Logica **84** (2006), pp. 277–322.
[5] Bolander, T. and P. Blackburn, *Termination for hybrid tableaus*, Journal of Logic and Computation **17** (2007), pp. 517–554.
[6] Bräuner, T., "Hybrid Logic and Its Proof-Theory," Springer, Heidelberg, 2011.
[7] Burgess, J., *Basic tense logic*, in: Gabbay and Guenthner, editors, *Handbook of Philosophical Logic, Volume 2*, Reidel, 1984 pp. 89–133.
[8] Cate, B. T., "Model Theory for Extended Modal Languages," Ph.D. thesis, ILLC, University of Amsterdam (2004).
[9] Kamp, H., *Formal Properties of 'Now'*, Theoria **37** (1971), pp. 237–273.
[10] Kaplan, D., *Demonstratives: An Essay on the Semantics, Logic, Metaphysics, and Epistemology of Demonstratives and Other Indexicals*, in: J. Almog, J. Perry and H. Wettstein, editors, *Themes from Kaplan*, Oxford University Press, Oxford, New York, 1989 pp. 481–564.

Appendix

Below are the hybrid tableau rules assumed in this paper; they are tense-logical versions of those introduced in Blackburn [3]. Here s, t, and u range over ordinary nominals and *now*, while a is used to indicate that a new (ordinary) nominal is being introduced. A formula ϕ is (logically) provable using this tableau system iff a closed tableau for $\neg @_i\phi$ can be constructed, where i is a nominal not occurring in ϕ. When axioms are introduced onto a tableau branch they are first prefixed by $@_i$, where i can be any nominal already occurring on that branch. While we hope this appendix is reasonably self-contained, some readers may find it useful to consult Blackburn [3], which contains several examples of tableau proofs, and a completeness proof for the minimal (modal) logic and all pure axiomatic extensions.

$$\frac{@_s\neg\phi}{\neg @_s\phi}\ [\neg] \qquad \frac{\neg @_s\neg\phi}{@_s\phi}\ [\neg\neg]$$

$$\frac{@_s(\phi\wedge\psi)}{\begin{array}{c}@_s\phi\\ @_s\psi\end{array}}\ [\wedge] \qquad \frac{\neg @_s(\phi\wedge\psi)}{\neg @_s\phi \mid \neg @_s\psi}\ [\neg\wedge]$$

$$\frac{@_s@_t\phi}{@_t\phi}\ [@] \qquad \frac{\neg @_s@_t\phi}{\neg @_t\phi}\ [\neg @]$$

$$\frac{@_sP\phi}{\begin{array}{c}@_sPa\\ @_a\phi\end{array}}\ [P] \qquad \frac{\neg @_sP\phi \quad @_sPt}{\neg @_t\phi}\ [\neg P]$$

$$\frac{@_sH\phi \quad @_sPt}{@_t\phi}\ [H] \qquad \frac{\neg @_sH\phi}{\begin{array}{c}@_sPa\\ \neg @_a\phi\end{array}}\ [\neg H]$$

$$\frac{@_sF\phi}{\begin{array}{c}@_sFa\\ @_a\phi\end{array}}\ [F] \qquad \frac{\neg @_sF\phi \quad @_sFt}{\neg @_t\phi}\ [\neg F]$$

$$\frac{@_sG\phi \quad @_sFt}{@_t\phi}\ [G] \qquad \frac{\neg @_sG\phi}{\begin{array}{c}@_sFa\\ \neg @_a\phi\end{array}}\ [\neg G]$$

$$\frac{@_sPt \quad @_tu}{@_sPu}\ [P\text{-Bridge}] \quad \frac{@_sFt \quad @_tu}{@_sFu}\ [F\text{-Bridge}] \quad \frac{@_sPt}{@_tFs}\ [P\text{-Trans}] \quad \frac{@_sFt}{@_tPs}\ [F\text{-Trans}]$$

$$\frac{[s \text{ on branch}]}{@_ss}\ [\text{Ref}] \qquad \frac{@_ts}{@_st}\ [\text{Sym}] \qquad \frac{@_st \quad @_t\phi}{@_s\phi}\ [\text{Nom}]$$

Here's a first example of the system in action. In this paper, when discussing *yesterday*, *today*, and *tomorrow* we introduced two axioms governing what we call One Step Alignment: $\mathit{today} \rightarrow G\neg\mathit{yesterday}$ and $\mathit{tomorrow} \rightarrow G\neg\mathit{today}$. Conspicuous by their absence are their backward-looking counterparts, namely $\mathit{today} \rightarrow H\neg\mathit{tomorrow}$ and $\mathit{yesterday} \rightarrow H\neg\mathit{today}$. But both are provable (by essentially identical proofs). Here's an example:

1	$\neg @_i(today \to H\neg tomorrow)$	
2	$@_i today$	$1, \neg\to$ *Elimination*
3	$\neg @_i H\neg tomorrow$	$1, \neg\to$ *Elimination*
4	$@_i Pj$	$3, \neg H$ *Elimination*
5	$\neg @_j \neg tomorrow$	$3, \neg H$ *Elimination*
6	$@_j tomorrow$	$5, \neg\neg$ *Elimination*
7	$@_j(tomorrow \to G\neg today)$	*One Step Alignment axiom*
8	$@_j G\neg today$	$6, 7,$ *Modus Ponens*
9	$@_j Fi$	$4, P$ *Transpose*
10	$@_i \neg today$	$8, 9,$ *G Propogation*
11	$\bot$	$2, 10$

Note that when we introduced the needed axiom at line 7, we prefixed it with an @-operator that already occurred on the branch. Also note that we applied Modus Ponens at line 8 rather than following strict tableau procedure and disjunctively splitting the branch; clearly this shortcut is harmless.

Let's look at a more interesting example. In this paper we introduced the Two Step Alignment axiom, $tomorrow \to G\neg yesterday$. We remarked that this axiom is superfluous when working with transitive models. The following tableau proof shows that, assuming transitivity, we can derive it using One Step Alignment and Now Placement:

1	$\neg @_i(tomorrow \to G\neg yesterday)$	
2	$@_i tomorrow$	$1, \neg\to$ *Elimination*
3	$\neg @_i G\neg yesterday$	$1, \neg\to$ *Elimination*
4	$@_i Fj$	$3, \neg G$ *Elimination*
5	$\neg @_j \neg yesterday$	$3, \neg G$ *Elimination*
6	$@_j yesterday$	$5, \neg\neg$ *Elimination*
7	$@_j(yesterday \to Fnow)$	*Now Placement axiom*
8	$@_j Fnow$	$6, 7,$ *Modus Ponens*
9	$@_j Fk$	$8, F$ *Elimination*
10	$@_k now$	$8,$ *F Elimination*
11	$@_k(now \to today)$	*Now Placement axiom*
12	$@_k today$	$10, 11,$ *Modus Ponens*
13	$@_i(tomorrow \to G\neg today)$	*One Step Alignment axiom*
14	$@_i G\neg today$	$12, 13,$ *Modus Ponens*
15	$@_i Fk$	$4, 9,$ *Transitivity Rule*
16	$@_k \neg today$	$14, 15,$ *G Propogation*
17	$\bot$	$12, 16$

One of the key points made in this paper is that we obtain complete proof systems for contextual validity simply by assuming *now* as an extra axiom at the start of tableau construction. More precisely, whereas to prove that ϕ is *logically* valid we attempt to construct a closed tableau starting with $\neg @_i\phi$; to prove that ϕ is *contextually* valid we attempt to construct a closed tableau starting with $@_i now$ and $\neg @_i\phi$ (in both cases, i is a nominal not occurring in ϕ). Here's a simple example. We shall show that *today* is contextually valid:

1	$@_i now$	
2	$\neg @_i today$	
3	$@_i(now \to today)$	*Now Placement axiom*
4	$@_i today$	$1, 3,$ *Modus Ponens*
5	$\bot$	$2, 4$