Contextual Validity in Hybrid Logic

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Abstract. Hybrid tense logic is an extension of Priorean tense logic in which it is possible to refer to times using special propositional symbols called nominals. Temporal indexicals are expressions such as *now*, *yesterday*, *today*, *tomorrow* and *four days ago* that have highly context-dependent interpretations. Moreover, such indexicals give rise to a special kind of validity—*contextual validity*—that interacts with ordinary logical validity in interesting and often unexpected ways. In this paper we model these interactions by combining standard techniques from hybrid logic with insights from the work of Hans Kamp and David Kaplan. We introduce a simple proof rule, which we call the Kamp Rule, and first we show that it is all we need to take us from logical validities involving *now* to contextual validities involving *now* too. We then go on to show that this deductive bridge is strong enough to carry us to contextual validities involving *yesterday*, *today* and *tomorrow* as well.

1 Introduction

Hybrid tense logic is an extension of Priorean tense logic in which it is possible to refer to times using special propositional symbols called nominals. Temporal indexicals are expressions such as *now*, *yesterday*, *today*, *tomorrow* and *four days ago*. The most obvious fact about temporal indexicals (and indeed, other indexicals such as *I*, *you*, and *here*) is that their interpretation is highly context-dependent. A less obvious fact about them is that they give rise to a new kind of validity—*contextual validity*—that interacts in interesting (and tricky) ways with logical validity. Modelling these interactions is a challenging task.

The logical study of temporal indexicals was initiated by Hans Kamp in his paper “Formal properties of ‘now’” [8]. This introduced several ideas—most notably, *two-dimensional semantics*—which have since become widely used in a number of fields. Kamp's work was refined and generalized to other indexicals by David Kaplan [9], who introduced the concept of *character*. The character of an indexical expression is a function specifying how the indexical exploits the context of utterance. For example, the character of *I* is a function which maps this indexical to the speaker in a given context, whereas the character of *you* maps this indexical to the person or people being addressed. We will specify characters for *now*, *yesterday*, *today*, and *tomorrow* later in this paper.

Both Kamp and Kaplan worked with ordinary tense logics. But, as has already been mentioned, there is a referential extension of tense logic called hybrid logic. Because hybrid logic allows reference to times, it seems natural to use it as the base logic for explorations of indexicals in the spirit of Kamp and Kaplan. After all, expressions such as *now*, *yesterday*, *today*, and *tomorrow* clearly do refer to certain (contextually selected) times, so why not work with a logic in which temporal reference is built in? The idea of using hybrid logic in this way dates back to Blackburn [1], and was explored in more depth by Blackburn and Jørgensen [3]. The latter paper gave complete tableau systems for hybrid reasoning with *now*, *yesterday*, *today*, and *tomorrow*, but it did something else which we think is more important: it showed that the indexical *now* acts as a sort of 'deductive bridge' between ordinary logical validity and contextual validity. This is rather surprising. It has been known ever since Kamp's pioneering work that the operator associated with 'now' is in a sense expressively weak. Nonetheless, in spite of its expressive weakness, 'now' is deductively important.

The present paper explores and clarifies this idea. We do so in two ways. First, we change the underlying semantics. In our previous paper, we used Kamp's original two-dimensional semantics for Now; here we shall use an (equivalent) semantics called *designated time semantics*. This is closer to the standard semantics of hybrid logic and is (we believe) more perspicuous. Second, we move from tableau-based deduction, to Hilbert-style axiomatic deduction. This may seem strange. Aren't tableaus easier to use than axiom systems? They certainly are—but in this paper we are not particularly interested in actually doing deductions. Rather, our goal is to clarify the inferential architecture, and axiom systems are a good way of doing that.

We proceed as follows. In Section 2 we introduce the basics of hybrid tense logic. In Section 3 we make an (almost invisible) extension, adding a new nominal *now* to the language. In Section 4 we introduce a standard axiomatization for hybrid tense logic and show that it is complete for the *now*-enriched language. At least, it's complete as far a *logical* validity is concerned, but what about *contextual* validity? Section 5 provides the answer. We introduce one more (very simple) rule which we call the Kamp Rule. The rule is unusual in that it can only be used once in any proof, and only as the very last step. Nonetheless, this rule is the bridge from the world of logical validity to the world of contextual validity. Moreover, as Section 6 shows, if we walk across this narrow bridge we will find the contextual logics of *yesterday*, *today*, and *tomorrow* waiting on the other side, as the Kamp Rule feeds a crucial piece of contextual information to the character functions of these indexicals. Section 7 concludes.

2 Hybrid Tense Logic

As already said, hybrid tense logic is a simple extension of ordinary Priorean tense logic in which it is possible to refer to times. It can do this because it contains a collection of special propositional symbols called *nominals*. Nominals are true at one and only one time: they 'name' the time they are true at. This is the framework we will use to explore temporal indexicals, so to get the ball rolling, let's define its syntax and semantics.

Let $\mathcal{L}$ be a standard minimal hybrid tense language: a set Nom of nominals, a set Prop of ordinary propositional symbols, boolean operators $\neg$ and $\wedge$, an $@_i$-operator for each nominal i, and two (existential) tense operators P and F. Formulas of $\mathcal{L}$ are built as follows:

$$\varphi ::= i \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid P\varphi \mid F\varphi \mid @_i\varphi.$$

We define $G\varphi$ to be $\neg F \neg\varphi$ and $H\varphi$ to be $\neg P \neg\varphi$ and say that G and F, and H and P, are dual operator pairs. Likewise, boolean symbols such as $\vee$, $\rightarrow$, $\leftrightarrow$ and $\perp$ are defined in the usual way. Note that a nominal i can occur syntactically in two distinct ways: in *formula position* as the atomic symbol i, or in *operator position* as in $@_i\varphi$. Finally, if a formula contains no ordinary propositional symbols, but only nominals as atomic symbols, it is a *pure formula*.

Models $\mathfrak{M}$ are based on frames (T, R). We think of T as a set of times and R as the earlier-later relation. What properties should R have? Well, we typically think of R as an irreflexive and transitive relation. But sometimes we think of it as a linear relation, and sometimes we think of it as branching towards the future. Moreover, for some applications we may want to think of R as dense, whereas for others we may need a discrete temporal order. And sometimes we want a first (or last) point of time, and sometimes we don't. Fortunately, we don't need to make such choices here: they are easy to specify axiomatically in hybrid logic (we'll discuss this later) so we don't need to hardwire them into the semantics. Thus we are free to work with an arbitrary relation R.

But to fully specify a model we also need an *information distribution* together with a *specification of names* for times of interest. Both tasks are performed by a valuation function V, which takes propositional symbols and nominals to subsets of points of T. Ordinary propositional symbols are unrestricted in their interpretation: they encode ordinary information, such as when it is raining, or when the printer was enabled, or when Felicity had her disastrous relationship with Brad. But we place an important restriction on the valuation $V(i)$ of any nominal i: this must be a *singleton* subset of T. This means (as we said above) that nominals enable us to specify names for times in T.

Given a model $\mathfrak{M} = (T, R, V)$ we define *satisfaction* as follows:

$\mathfrak{M}, t \models a$	iff	a is atomic and $t \in V(a)$
$\mathfrak{M}, t \models \neg\varphi$	iff	$\mathfrak{M}, t \not\models \varphi$
$\mathfrak{M}, t \models \varphi \wedge \psi$	iff	$\mathfrak{M}, t \models \varphi$ and $\mathfrak{M}, t \models \psi$
$\mathfrak{M}, t \models P\varphi$	iff	for some t', $t'Rt$ and $\mathfrak{M}, t' \models \varphi$
$\mathfrak{M}, t \models F\varphi$	iff	for some t', tRt' and $\mathfrak{M}, t' \models \varphi$
$\mathfrak{M}, t \models @_i\varphi$	iff	$\mathfrak{M}, t' \models \varphi$ and $t' \in V(i)$.

Most of this is familiar from ordinary Priorean tense logic. In particular, $F\varphi$ scans the future looking for a time where φ is true (thus it makes an existential claim about the future) while its dual form, $G\varphi$, claims that φ is going to be true at all future times (a universal claim). Analogously, $P\varphi$ scans the past looking for a φ-verifying time, while $H\varphi$ claims that φ has always been true in the past.

What is new is the role played by the nominals and the @-operators. First, note that an atom a can be either a nominal or a propositional symbol, so the first clause of the definition handles both types of symbol in a uniform way. It also means that our fundamental restriction on the interpretation of nominals is built right into the heart of the satisfaction definition. Next, note that $@_i\varphi$ is satisfied at a time in a model $\mathfrak{M}$ if and only if φ is satisfied at the time that i names in $\mathfrak{M}$. So to speak, $@_i\varphi$ peeks at the time named i (and there *must* be such a time because of the restriction imposed on the interpretation of nominals) and checks whether φ is satisfied then or not. Note also that a formula of the form $@_i\varphi$ is satisfied at the time named i in $\mathfrak{M}$ if and only if it is satisfied at *all* times in $\mathfrak{M}$; this is because all that is relevant for formulas of this form is whether φ is satisfied at the point named i or not.

We say that a formula φ is *true in a model* $\mathfrak{M}$ if and only if it is satisfied at all times in $\mathfrak{M}$, and we say that φ is *logically valid* if and only if it is true in all models. Some examples of logical validity may be helpful: the propositional tautology $p \vee \neg p$ is (obviously) logically valid, as is the ordinary Priorean tense logical formula $Fp \vee Fq \rightarrow F(p \vee q)$, which simply says that if p is true in the future or q is true in the future then $p \vee q$ is true in the future. More interestingly, here's a genuinely *hybrid* tense logical validity: it contains an ordinary propositional symbol p and a nominal i in both formula and operator position:

$$Fi \wedge @_i p \rightarrow Fp.$$

This says that if the point named i lies in the future, and p is true at the point named i, then p will be true in the future. Intuitively, this should be logically valid, and indeed its validity follows from the definitions just given.

That's all we need to know about hybrid tense logic for the moment, so let's turn to the central task of the paper: the modelling of temporal indexicality.

3 Adding *now*

For a start, we will just add the temporal indexical *now* to our language. This will be the most straightforward addition we shall make—we're pretty much going to treat *now* as a nominal—but it will turn out to be the most fundamental. As we shall see, *now* is a key that will let us unlock the contextual semantics of the temporal indexicals *yesterday*, *today*, and *tomorrow*. By the end of the paper it will be clear that although *now* is a nominal, it is not 'just another' nominal.

And so to work. We first add the new atomic symbol *now* to $\mathcal{L}$, thus obtaining the language $\mathcal{L}(now)$. Syntactically, *now* is simply a nominal. Like ordinary nominals, *now* can occur in formula position as the atomic symbol *now*, and in operator position, as in $@_{now}\varphi$. Indeed, this latter expression is simply our hybrid-logical reconstruction of Hans Kamp's [8] celebrated Now operator.

But what is its semantics? The idea we shall use here is simplicity itself: take an ordinary model $\mathfrak{M} = (T, R, V)$ for hybrid tense logic and choose one of its times (that is, an element of T) as the *designated time*. Later in the paper, when we model other temporal indexicals and introduce character functions, we shall think of the designated time as the "utterance time of the context associated with the model". But here we just think of the designated time as the now of the model, and insist that our new atomic symbol *now* names now.

Spelling this out precisely, a *designated time model* $\mathfrak{M} = (T, R, V, t_0)$ is an ordinary model $\mathfrak{M}' = (T, R, V')$, together with a designated time $t_0 \in T$, where V is V' extended in the following way:

$$V(a) = \begin{cases} \{t_0\}, & \text{if } a \text{ is } now, \\ V'(a), & \text{otherwise.} \end{cases}$$

So the fact that *now* denotes the designated time—that is, that *now* really does mean now—is hardwired into the definition of what valuations are.[1]

Given the concept of a designated time model $\mathfrak{M} = (T, R, V, t_0)$, the satisfaction definition for $\mathcal{L}(now)$ is a straightforward extension of the one given earlier for hybrid tense logic. Indeed, to the earlier given clauses we simply add:

$$\begin{array}{lll} \mathfrak{M}, t \models now & \text{iff} & t \in V(now) \\ \mathfrak{M}, t \models @_{now}\varphi & \text{iff} & \mathfrak{M}, t' \models \varphi \text{ and } t' \in V(now). \end{array}$$

Because the special role played by the designated time t_0 is built into the definition of the valuation V, these clauses (which have exactly the same form as the clauses for ordinary nominals) guarantee that *now* really is a name for t_0, and that $@_{now}$ really is a hybrid-logical reconstruction of Kamp's Now operator.

We are ready for an idea that has underpinned the study of indexical expressions since the pioneering work of Hans Kamp and David Kaplan: *indexicals are interesting because they give rise to a new species of validity.* As before, we have the familiar notion of logical validity, and indeed this is defined for $\mathcal{L}(now)$ in the same manner as it was for $\mathcal{L}$. That is, a formula φ is *true in a designated time model* $\mathfrak{M}$ if and only if it is satisfied at all times in $\mathfrak{M}$, and φ is *logically valid* when it is true in all designated time models.

But indexicals introduce a second notion of validity, which we call contextual validity. A formula φ is *contextually true* in a designated time model $\mathfrak{M}$ if and only if it is satisfied *at the designated point* t_0 *of* $\mathfrak{M}$. That is, contextual truth in $\mathfrak{M}$ means that $\mathfrak{M}, t_0 \models \varphi$. And now for the crucial definition: a formula φ is *contextually valid* when it is contextually true in all designated time models. In words: a contextual validity is a formula that is true at the now of every model.

[1] Kamp's classic "Formal properties of 'now'" [8] uses a different semantics: it uses (indeed it introduced) the idea of *two-dimensional semantics* in which formulas are evaluated at *pairs* of times. But the approach we are using in this paper, which is sometimes called *pointed semantics*, also has a long history; for example, it was used by John Burgess [6] in his 1984 survey of tense logic when discussing Kamp's work. Moreover, pointed semantics is generally the preferred option in contemporary discussions of the Actuality operator, a modal operator that picks out the actual world in much the same way that the Now operator selects the utterance time; see Blackburn and Marx [4] for discussion and results. It would be a mistake to exaggerate the differences between the two approaches (for the simple propositional systems discussed here they are equivalent) and indeed our earlier work on temporal indexicals (see Blackburn and Jørgensen [3]) used Kaplan's generalisation of Kamp's original two-dimensional semantics. Nonetheless, we find the approach used here more perspicuous, both technically and conceptually.

Contextual validity is central to this paper, so let's consider some examples. As discussed earlier, propositional tautologies like $p \vee \neg p$, are logically valid, as are more complex formulas like $Fp \vee Fq \rightarrow F(p \vee q)$ and $Fi \wedge @_i p \rightarrow Fp$. To the point, *logically* valid formulas are *contextually* valid too. Why? Well, logical valid means "satisfied at *all* points in *all* designated time models"—hence any logical validity must be satisfied at the designated time in any designated model. In short, the set of logical validities is a subset of the set of contextual validities.

But it is a *proper* subset. That is, there are contextual validities that are not logical validities. To give the simplest example, *now* is not logically valid, but it is contextually valid: given any $\mathfrak{M}$ we have that *now* is satisfied at the designated point t_0. This is for the obvious reason that *now* is hardwired to denote the designated point, and so for all models $\mathfrak{M}$ we have $\mathfrak{M}, t_0 \models now$.

Here's another example, one that will play a suggestive role in our later work: the formula-schema $\varphi \leftrightarrow @_{now}\varphi$ is not logically valid, but it is contextually valid. Why is it not logically valid? Well, suppose we are working in a model in which *now* denotes the time you are reading these words (yes, right now, here in the 21st century!) and p means "Jane Austin is writing the last words of *Persuasion*". Well, if we look back in time to the moment in the early 19th century when Ms Austin finished her masterpiece, p certainly was true. But at that historic moment, $@_{now}p$ was clearly false: after all, this formula says she finished her masterpiece right now, that is, in the 21st century! Hence $p \leftrightarrow @_{now}p$ was false at an important moment of English literary history. So we have falsified an instance of the schema, and hence it is not logically valid.

But it *is* contextually valid. For suppose we evaluate any given φ at the designated time of some model $\mathfrak{M}$. Regardless what proposition φ is, it will be either true or false then. But then $@_{now}\varphi$ will have the same truth value as φ, for the simple reason that that *now* picks out the designated point, and $@_{now}\varphi$ reports the truth value of φ at that special time. To put it another way: when evaluating any formula φ *at* the designated point of any model, φ and $@_{now}\varphi$ stand or fall together. But this means that $\varphi \leftrightarrow @_{now}\varphi$ is a contextual validity.

4 Axiomatizing Logical Validity

In the previous section we defined the syntax and semantics of the language $\mathcal{L}(now)$. We defined two notions of validity for the language, and saw they were distinct. And this leads to some obvious questions. Can we characterize these two different logics? In particular, can we axiomatize them? And can we axiomatize them in a simple fashion that show the connection between them?

We are going to do this, and we are going to do it in two steps. In this section, we shall show that logical validity for $\mathcal{L}(now)$ can be reduced to ordinary hybrid tense logical validity, and hence that standard hybrid axiom systems successfully capture this notion. We postpone till the following section the trickier issue of capturing contextual validity axiomatically.

Here's the axiom system we shall work with. When working with $\mathcal{L}(now)$ we call it $K_h^t(now)$, and then a and b in the axioms listed in Figure 1 range over both ordinary nominals and the *now* nominal. When working with $\mathcal{L}$, we call this system K_h^t, and then a and b range over ordinary nominals.[2] That is, $K_h^t(now)$ and K_h^t differ only in whether *now* is in the language or not.

The system $K_h^t(now)$

Axioms

CT	All classical tautologies	
Duality	$\vdash Pp \leftrightarrow \neg H \neg p$	$\vdash Fp \leftrightarrow \neg G \neg p$
$K_\Box$	$\vdash H(p \to q) \to (Hp \to Hq)$	$\vdash G(p \to q) \to (Gp \to Gq)$
$K_@$	$\vdash @_a(p \to q) \to (@_a p \to @_a q)$	
$\text{Selfdual}_@$	$\vdash @_a p \leftrightarrow \neg @_a \neg p$	
$\text{Ref}_@$	$\vdash @_a a$	
Agree	$\vdash @_a @_b p \leftrightarrow @_b p$	
Intro	$\vdash a \to (p \leftrightarrow @_a p)$	
$\text{Back}_{P,F}$	$\vdash P @_a p \to @_a p$	$\vdash F @_a p \to @_a p$
Interact	$\vdash @_a Pb \leftrightarrow @_b Fa$	

Rules

MP	If $\vdash \psi \to \varphi$ and $\vdash \psi$ then $\vdash \varphi$	
Subst	If $\vdash \varphi$ then $\vdash \varphi^\sigma$	
$\text{Gen}_{H,G}$	If $\vdash \varphi$ then $\vdash H\varphi$	If $\vdash \varphi$ then $\vdash G\varphi$
$\text{Gen}_@$	If $\vdash \varphi$ then $\vdash @_a \varphi$	
Name	If $\vdash @_a \varphi$ and a does not occur in φ then $\vdash \varphi$	
BG_P	If $\vdash @_a Pb \to @_b \varphi$ and $b \neq a$ does not occur in φ then $\vdash @_a H\varphi$	
BG_F	If $\vdash @_a Fb \to @_b \varphi$ and $b \neq a$ does not occur in φ then $\vdash @_a G\varphi$	

Fig. 1.

Two general remarks are in order. First, when it comes to dealing with *now*, there is nothing particularly special about the axiomatization that we have chosen. Indeed, the whole point of the (essentially semantic) argument we shall soon give is that logical validity for $\mathcal{L}(now)$ is reducible to logical validity for $\mathcal{L}$, that is, to ordinary hybrid tense logical validity. In effect, we show that *any* sound

[2] In fact, K_h^t is just the tense-logical version of a complete axiomatization of the minimal hybrid modal logic given in Blackburn and ten Cate [2]. While the details of K_h^t and $K_h^t(now)$ don't play an important role in this paper, we would like to make a remark about the substitution rule being used: σ is any substitution that uniformly substitutes formulas in $\mathsf{Nom} \cup \{now\}$ by formulas in $\mathsf{Nom} \cup \{now\}$, and uniformly substitutes ordinary propositional symbols by arbitrary formulas.

and complete axiomatization of logical validity in $\mathcal{L}$ captures logical validity for $\mathcal{L}(now)$ as well. We chose this axiomatization because we know it and like it.

Second, to return to a remark made earlier, when working with real applications, we often want to put restrictions on the properties possessed by the relation R. For example, we may wish to work with branching time or linear time, with dense time or discrete time. We remarked that hybrid logic made it easy to impose restrictions on the flow of time axiomatically, and this was no idle boast. One of the most useful aspects of hybrid logic is its deductive modularity.

Here's a simple example. Consider the following three axioms. A little thought shows that they correspond to irreflexivity, transitivity and linearity respectively:

$$@_i \neg Fi \qquad FFi \rightarrow Fi \qquad @_i Fj \vee @_i j \vee @_j Fi$$

For example, the formula on the left says that if you are at the point named i, you cannot look into the future and see i, which is a way of describing irreflexivity. Adding these three axioms gives us a sound and complete proof system when time possesses these three properties, and this example is only the tip of a very large iceberg. Recall that a pure formula is a formula that only contains nominals as atoms. A fundamental result of hybrid logic tells us that when we add additional pure axioms (note that the three axioms in our example above are pure) then the resulting system is guaranteed to be complete with respect to models the axioms describe.[3] And because of our strategy of reducing $\mathcal{L}(now)$ logical validity to $\mathcal{L}$ logical validity, this deductive modularity will be inherited by all our indexical logics. This is one of the reasons we feel that hybrid logic is a particularly good logical setting for exploring temporal indexicals.

Time to return to our axiomatic work. First we check soundness:

Theorem 1 (Soundness). *The axioms and rules denoted by $K_h^t(now)$ are sound with respect to designated time models.*

Proof. The proof is a straightforward variant of the ordinary inductive soundness proof for hybrid tense logics.

Now for the key lemma. That the axiom system $K_h^t(now)$ characterises the logically valid formulas follows from the observation that satisfiability of formulas in $\mathcal{L}(now)$ can be reduced to satisfiability of formulas in $\mathcal{L}$. We'll use the following notation: if in φ we uniformly substitute ρ for ψ we obtain $\varphi[\psi \leftarrow \rho]$.

Lemma 1 (Reduction to Basic Hybrid Tense Logic). *Let φ be a formula in $\mathcal{L}(now)$ and j a nominal not occurring in φ, then $\varphi[now \leftarrow j]$ is satisfiable in an ordinary model iff φ satisfiable is in a designated time model.*

[3] It would take us too far from the concerns of this paper to discuss why hybrid logic is deductively modular, but the two more complex rules, BG_P and BG_F, play a central role here. For a discussion of the role of such rules, see Chapter 7, Section 3 of Blackburn, De Rijke and Venema [5] and Blackburn and ten Cate [2]. For detailed model-theoretic results on what can be achieved using pure axioms, see ten Cate [7].

Proof. Suppose some φ in $\mathcal{L}(now)$ is given with j not occurring in φ. We prove by induction on φ a slightly stronger version of the lemma, namely that $\varphi[now \leftarrow j]$ is satisfied at t in the ordinary model $\mathfrak{M} = (T, R, V)$ iff φ is satisfied at t in the designated time model $\mathfrak{M}' = (T, R, V', V'(j))$. Here V' is identical with V on all nominals and propositional symbols and $V'(now) = V(j)$. Note our abuse of notation: we actually mean the unique element of $V'(j)$ when we write the fourth element of the designated time model tuple—we use this conflation systematically in the proof below. Also, note that $V'(j) = V(j)$. So $V'(now)$, $V'(j)$, and $V(j)$ are alternative ways of picking out the designated time.

First, the three base cases. Suppose φ is i (which is the same as $\varphi[now \leftarrow j]$) and that it is satisfied at t in $\mathfrak{M} = (T, R, V)$. Let $\mathfrak{M}' = (T, R, V', V'(j))$ be the designated time model defined as just described. Clearly $\mathfrak{M}, t \models i$ iff $V(i) = t$ iff $V'(i) = t$ iff $\mathfrak{M}', t \models i$. This completes the argument for ordinary nominals. And clearly, if φ is p, then an analogous argument also works. So we only need to check the case when φ is *now*. So suppose that $\varphi[now \leftarrow j]$, which is j, is satisfied at t in $\mathfrak{M} = (T, R, V)$. Then for the designated time model $\mathfrak{M}' = (T, R, V', V'(j))$, we have $\mathfrak{M}', t \models now$. As for the other direction, if t is the denotation of both j and *now* in $\mathfrak{M}'$, then t is the denotation of j under V in $\mathfrak{M}$. This completes the three base cases.

We shall prove one case of the inductive step of the argument. Let φ be $@_{now}\psi$. Then $\varphi[now \leftarrow j]$ is $@_j\psi[now \leftarrow j]$. Suppose this is satisfied at t in $\mathfrak{M} = (T, R, V)$. If t' is the unique element of $V(j)$, then $\mathfrak{M}, t' \models \psi[now \leftarrow j]$. For $\mathfrak{M}' = (T, R, V', V'(j))$ as defined above, the induction hypothesis gives us that $\mathfrak{M}', t' \models \psi$, and as t' is the denotation of *now* under V' we have that $\mathfrak{M}', t \models @_{now}\psi$. The other direction is similar, as are the rest of the inductive cases. This completes the proof.

Theorem 2. (Logical Completeness) $K_h^t(now)$ *is complete with respect to designated time models. Moreover, when pure formulas are added as additional axioms, it is complete with respect to the class of models they define.*

Proof. Recall that K_h^t is a complete axiomatisation of hybrid tense logic in the *now*-free language $\mathcal{L}$. Let φ be a formula of $\mathcal{L}(now)$ that is $K_h^t(now)$-consistent, and let j be a nominal not occurring in φ. Then $\varphi[now \leftarrow j]$ is a formula in $\mathcal{L}$, and it must be K_h^t-consistent—for if it wasn't, we could prove the inconsistency of φ in $\mathcal{L}(now)$, as *now* functions syntactically like any other nominal. Therefore, by the completeness of K_h^t, we know that $\varphi[now \leftarrow j]$ has a model. By our reduction to hybrid tense logic (Lemma 1) this means that φ has a designated time model, which means that $K_h^t(now)$ is complete with respect to the designated model semantics, as claimed. That adding pure formulas as additional axioms yields additional completeness results is standard in hybrid logic (recall the discussion of deductive modularity).

5 Axiomatizing Contextual Validity

Now we want to axiomatize *contextual* validity, and indeed, to axiomatize it as an extension of our previous axiomatisation, $K_h^t(now)$, which captured *logical* validity in $\mathcal{L}(now)$. How are we to do this? The answer is both surprisingly simple and rather subtle. First the simplicity: all we have to do is extend $K_h^t(now)$ with the following rule, which we have called the Kamp Rule:[4]

> **Kamp Rule:** If we have proved $@_{now}\varphi$, then we have a proof of φ. That is:
> If $\vdash @_{now}\varphi$ then $\vdash \varphi$.
> **Restriction**: Can only be used once in a proof and only as the very last step.

This rule is *contextually sound*. For let any $\mathfrak{M} = (T, R, V, t_0)$ be given, and suppose $@_{now}\varphi$ is satisfied at the designated time. That is, suppose we have $\mathfrak{M}, t_0 \models @_{now}\varphi$. This means that $\mathfrak{M}, t_0 \models \varphi$. So the conclusion of the Kamp Rule is satisfied in the same model at the same (designated) time, and thus the rule is contextually sound.

Here's a simple example of the rule at work: a two-step proof of *now*:

1.	$@_{now}now$	(Standard axiom, instance of $@_i i$)
2.	now	(Kamp Rule)

This makes good sense. As we saw earlier, *now* is the simplest example of a contextual validity, and so it should be provable in any complete system for contextual validity.

But now for the subtlety. Why did we impose the restriction that the rule can only be used once, and only as the very last step of the proof? Well, for the simple reason that without this restriction the system would collapse! Why is this? Because, as we mentioned at the start of the paper, logical and contextual validity interact in tricky ways. Let's think this through.

Suppose φ is logically valid. That is, by the previous completeness result, φ is provable in $K_h^t(now)$. So we have $\vdash \varphi$, hence by using the Gen_G rule we can obtain $\vdash G\varphi$. And this makes perfect sense: if φ is *logically* valid then of course it is going to be true at all future times, hence $G\varphi$ is also a logical validity, and thus it should be provable. No problem here. It's exactly what we want.

But now suppose we add the Kamp Rule *without* the restriction. Well, we have just given a two line proof of *now*, so we have $\vdash now$. And here comes the collapse: we now use Gen_G to prove that $\vdash Gnow$, which means that it is always going to be the case that *now*. In terms of our models this means that all future points are identical to the designated time, and that is not what we want at all. Therefore, we can only apply the Kamp Rule once in a proof—and then stop!

But then, what about completeness? With such a drastic restriction in place, surely the rule is too weak to yield contextual completeness? But it's not: with

[4] As far as we are aware, this rule has not been proposed before. We call it the Kamp Rule because it trades on ideas similar to those Kamp used in his proof that his Now operator is, in certain sense, eliminable in standard propositional tense logic. For Kamp's original proof of the elimination result, see [8], and for a hybrid logic generalization, see Blackburn and Jørgensen [3].

the help of the following lemma we shall prove the contextual completeness of $K_h^t(now) + \mathrm{KR}$.

Lemma 2. *For $\varphi \in \mathcal{L}(now)$, $@_{now}\varphi$ is logically valid iff φ is contextually valid.*

Proof. Suppose $@_{now}\varphi$ is logically valid. Let $\mathfrak{M} = (T, R, V, t_0)$ be given. We need to show that φ is contextually true in $\mathfrak{M}$, that is, that it is satisfied at the designated point t_0. As $@_{now}\varphi$ is logically valid, for all times t in T we have that $\mathfrak{M}, t \models @_{now}\varphi$. But this means that $\mathfrak{M}, t_0 \models \varphi$. For the other direction, suppose φ is contextually valid, that is, satisfied in any model at the designated point. Given any $\mathfrak{M} = (T, R, V, t_0)$ we need to show that $@_{now}\varphi$ is satisfied at any $t \in T$. But this is clear: by assumption we have that $\mathfrak{M}, t_0 \models \varphi$. This means, for all $t \in T$ we have that $\mathfrak{M}, t \models @_{now}\varphi$.

Theorem 3 (Contextual Completeness). $K_h^t(now) + \mathrm{KR}$ *is contextually complete with respect to designated time models. Moreover, when pure formulas are added as additional axioms, it is contextually complete with respect to the class of models they define*

Proof. If φ is contextually valid, then, by the previous lemma, $@_{now}\varphi$ is logically valid. Hence, by our previous completeness theorem, we have that $@_{now}\varphi$ is provable in $K_h^t(now)$. Simply take this proof and apply the Kamp Rule to the end formula: this gives us the required proof of φ. The result about the effect of additional pure axioms is standard in hybrid logic.

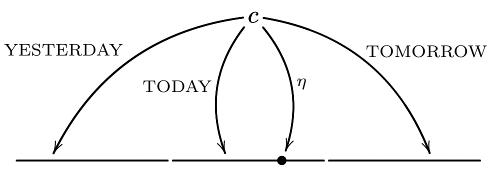
The moral of the story is this. Yes, logical and contextual validity interact in tricky ways. But these effects can be unravelled, even in a Hilbert system. In particular, this completeness result tells us is that any axiomatic proof of a contextually valid formula φ can be broken down into a (possibly very lengthy) proof of $@_{now}\varphi$, followed by a one step application of the Kamp Rule which strips off the outermost operator.

There is an important point of contact between the use of the the Kamp Rule and the tableau system for contextual validity developed in Blackburn and Jørgensen [3]. In our earlier paper, logical validity was captured using a standard hybrid tableau system. Contextual validity was captured by building tableaus for the input formula in which the root node of the tableau was labelled *now*. In other words, capturing contextual validity tableau-style means that instead of trying to falsify the input formula at an *arbitrary* time, you have to try to falsify it at the time where *now* is true. Labeling the root node of the tableau with *now*, which happens as the very *first* step of tableau construction, corresponds to the use of the Kamp Rule as the very *last* step of a Hilbert-style proof.

6 Crossing to *yesterday*, *today* and *tomorrow*

It is time to consider other temporal indexicals. Accordingly, we enrich $\mathcal{L}(now)$ with three new propositional symbols: *yesterday*, *today* and *tomorrow*. Like *now*, all three symbols can occur in formula position. Unlike *now*, they cannot occur in operator position. This is because they are not nominals, and @ requires nominals as subscripts.

Well, if they are not nominals, then what are they? Simply three special propositional symbols mutually constrained in their interpretation, but not constrained (as nominals are) to be true at a single time. Intuitively (and unsurprisingly) each of these symbols represents a day. The following diagram illustrates how to envisage the mutual constraints on their interpretations:



That is, each of our three new symbols denotes a "daylike" set of times, each correctly positioned in the model with respect to the others, and with respect to the designated time t_0, which is marked in the diagram as a black dot.

In fact, the above diagram is essentially a pictorial representation of the *characters* of the indexicals *yesterday*, *today*, *tomorrow* and *now*. As we said earlier, a character function stipulates how an indexical exploits the context. In this diagram we see a context c and its image under four character functions, YESTERDAY, TODAY, TOMORROW and η. Intuitively, η is the most fundamental: $\eta(c)$ is the utterance time of c, the time that *now* names. The sets of points picked out by the other indexicals group naturally around this central time.

Models for our expanded language simply build in this extra structure. First, instead of designated time models, we now work with *designated context models*. These are simply 4-tuples $\mathfrak{M} = (T, R, V, c)$ where $\mathfrak{M} = (T, R, V)$ is a model for hybrid tense logic, and c is the designated context.

It only remains to specify the valuation functions for our three new symbols and for *now* in this new setting. And (by this stage) the reasons for the following stipulations should be clear. If V' is a valuation for hybrid tense logic on a model $\mathfrak{M}$, and c is a context in $\mathfrak{M}$, then we extend V' to a valuation V for our enriched language as follows:

$$V(a) = \begin{cases} \{\eta(c)\}, \text{ if } a \text{ is } \textit{now}, \\ \textsc{yesterday}(c), \text{ if } a \text{ is } \textit{yesterday}, \\ \textsc{today}(c), \text{ if } a \text{ is } \textit{today}, \\ \textsc{tomorrow}(c), \text{ if } a \text{ is } \textit{tomorrow}, \\ V'(a), \text{ otherwise.} \end{cases}$$

Once more, we have hardwired the meaning of our special symbols at the atomic level, and because of this we can simply interpret the language as before.

But we are not yet finished. The previous diagram shows a well-behaved context, with well-behaved character functions. That is, everything lines up in the expected way. But if we want a complete logic for working with our new symbols, we must pin down what it is about the previous diagram that we like. And this is easy to do. We simply stipulate that we will only work with models in which the following axioms are true at all times:

Now Placement	**Disjointness**
$now \to today$	$today \to \neg tomorrow$
$yesterday \to F now$	$today \to \neg yesterday$
$tomorrow \to P now$	$yesterday \to \neg tomorrow$
One Step Alignment	**Two Step Alignment**
$today \to G \neg yesterday$	
$tomorrow \to G \neg today$	$tomorrow \to G \neg yesterday$
Convexity	**No Gaps**
$P yesterday \wedge F yesterday \to yesterday$	$P yesterday \wedge F today \to yesterday \vee today$
$P today \wedge F today \to today$	$P today \wedge F tomorrow \to today \vee tomorrow$
$P tomorrow \wedge F tomorrow \to tomorrow$	

Suppose all these axioms are true in some designated context model $\mathfrak{M}$ (that is, true at all times t in $\mathfrak{M}$). Then it is easy to see that at $\eta(c)$—the utterance time—there will be a yesterday, a today, and a tomorrow, and that these will be grouped around $\eta(c)$ exactly as in our picture.

But are they complete? Well, logical completeness is clear. By definition, we are only going to work with models that make the above axioms globally true. Hence (by definition) these axioms are complete with respect to the desired class of models. It's when we get to *contextual* completeness that things become more interesting. That's when we start bridge building.

Look at the *form* of these axioms. Imagine you are in the context of utterance. Here, of course, *now* is true. But this means modus ponens fires, making *today* true (this is due to the first Now Placement axiom). And indeed, *all* the logical consequences of *now* are going to hold, and all the logical consequences of these axioms will hold, and the familiar properties of our four indexicals just drop into place. Basically, the axioms given above record general properties of the four character functions. And when this information is relevant—that is, when we are reasoning about the utterance time—we access it by contextual reasoning.

And this is exactly what the Kamp Rule lets us do proof-theoretically. Consider the following (simplified) Hilbert proof:

$$\dfrac{\dfrac{@_{now} now \qquad \dfrac{now \to today}{@_{now}(now \to today)}\,(\mathrm{Gen}_{@})}{@_{now} today}\,(\mathrm{MP})}{today}\,(\mathrm{KR})$$

The second-to-last line of the proof is a logical truth, namely $@_{now} today$. If we make use of the Kamp Rule at this point—that is, if we walk across the bridge and say: *I really am here now!*—then we strip of the outer operator and realize that (right then and there!) we are in the day called today. As we said at the start of the paper: the Kamp Rule feeds a crucial piece of information to the other indexicals. And that information is simply: *Now!*

7 Conclusion

In this paper we argued that hybrid logic was a good setting for exploring temporal indexicality. The technical arguments in favour of hybrid logic are strong: it is deductively modular, well understood, and the fact that temporal reference is built into its very core makes it a natural candidate for this application.

But the heart of this paper was conceptual, not technical. We wanted to show that (at least for temporal indexicals) the path from logical validity to contextual validity is unexpectedly simple: the indexical *now* provides a bridge to the contextual validity of other indexicals. And this leads to our our next question: what happens when we move beyond temporal indexicals to the full range of indexicals considered by David Kaplan? We don't expect *now* to provide a bridge to non-temporal indexicals such as *you* and *here*, but are there other bridge indexicals? And are there analogs of the Kamp Rule? And can hybrid logic yield perspicuous analyses in these richer setting? We hope to find out.

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